ArithmeticSite

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Chapter 1

The multiplicative monoid $\mathbb{N}^\times$

The Arithmetic Site, introduced by Connes and Consani, is a geometric object — a topos equipped with a structure sheaf — whose points in the topos-theoretic sense are in natural bijection with the adèle class space of $\mathbb{Q}$. Its construction begins with a single, elementary monoid: the positive integers under multiplication. Everything else — the tropical semiring, the presheaf topos, the structure sheaf, and ultimately the connection to the Riemann zeta function — is built on top of this one piece of data.

Definition 1.1. Let $\mathbb{N}^\times$ denote the set $\{1, 2, 3, ...\}$ of strictly positive integers equipped with ordinary multiplication. This is a commutative monoid with unit 1. In Lean we represent this as `PNat` with its multiplicative structure, via the abbreviation `NPos`.

The unique factorisation theorem says that $\mathbb{N}^\times$ is the free commutative monoid on the set of primes: every element factors uniquely as a finite product $\prod_p p^{a_p}$. This arithmetic richness — the interplay of primes, divisibility, and factorisation — is what makes $\mathbb{N}^\times$ the right monoid for the Arithmetic Site. As we will see, it is ultimately responsible for the fact that the points of $\widehat{\mathbb{N}^\times}$ are parametrised by the adèle class space of $\mathbb{Q}$, and that the Hasse–Weil zeta function of the site recovers the Riemann zeta function.

Definition 1.2. The *classifying category* $B\mathbb{N}^\times$ of $\mathbb{N}^\times$ is the one-object category whose single object is $*$ and whose morphisms $* \to *$ are the elements of $\mathbb{N}^\times$, with composition given by multiplication.

Remark 1.3. The passage from a monoid M to the one-object category BM is the standard *delooping* construction. A presheaf on BM — that is, a functor $BM^{\mathrm{op}} \to \mathbf{Set}$ — is exactly a set X equipped with a left action of M: each element $n \in M$ acts as a function $X \to X$, and the action is compatible with composition. (Left and right actions coincide here since $\mathbb{N}^\times$ is commutative.) The presheaf topos $\widehat{\mathbb{N}^\times}$ introduced in Chapter 3 is precisely the category of such sets-with-$\mathbb{N}^\times$-action, and it is in this topos that the Arithmetic Site lives.

Chapter 2

The tropical semiring $\bar{\mathbb{N}}$

The tropical semiring $\bar{\mathbb{N}}$ is the second fundamental ingredient of the Arithmetic Site. It plays the role of the *structure sheaf*: as Connes and Consani put it, $\bar{\mathbb{N}}$ is the semiring on which $\mathbb{N}^\times$ acts by multiplication, and it is this action that makes $\bar{\mathbb{N}}$ into a sheaf of semirings over the topos $\widehat{B\mathbb{N}^\times}$. The arithmetic of $\bar{\mathbb{N}}$ is characteristic-1 arithmetic — addition is idempotent — which is the tropical, or min-plus, shadow of ordinary arithmetic. Connes and Consani define $\bar{\mathbb{N}}$ explicitly as $(\mathbb{N} \cup \{\infty\}, \inf, +)$, confirming the min-plus convention: addition is min, with $+\infty$ as the additive identity. When elements of $\bar{\mathbb{N}}$ are written as exponents q^x for $q \in (0,1)$, the addition becomes max on values (since smaller exponent means larger value), which is why the stalks in the paper appear in max-plus form as $[0,1]_{\max}$ and $\mathbb{R}_{\max}$.

Definition 2.1. The *tropical semiring* $\bar{\mathbb{N}}$ is the set $\mathbb{N} \cup \{+\infty\}$ equipped with:

- *Tropical addition*: $a \oplus b := \min(a, b)$, with identity element $+\infty$.
- *Tropical multiplication*: $a \odot b := a + b$ (ordinary addition), with identity element 0.

In Lean, $\bar{\mathbb{N}}$ is represented as `Tropical (WithTop ℕ)`, via the abbreviation `NBar`. The zero element is `trop ⊤` (i.e. $+\infty$) and the multiplicative identity is `trop 0`.

Lemma 2.2. *$\bar{\mathbb{N}}$ is a commutative semiring.*

Proof. This is Mathlib's canonical `CommSemiring` instance for the `Tropical` type over `WithTop ℕ`. ☐

Lemma 2.3. *Tropical addition on $\bar{\mathbb{N}}$ is idempotent: $a \oplus a = a$ for all $a \in \bar{\mathbb{N}}$.*

Proof. $a \oplus a = \min(a, a) = a$. In Lean this follows from `Tropical.add_eq_left`. ☐

Remark 2.4. Idempotency of addition — equivalently, the identity $1 \oplus 1 = 1$ — is the defining feature of *characteristic* 1 arithmetic. It is what places $\bar{\mathbb{N}}$ in the same conceptual family as finite fields $\mathbb{F}_p$ (characteristic p), with min playing the role that $+ \bmod p$ plays there. The scaling endomorphisms defined below are the characteristic-1 analogue of the Frobenius endomorphism $x \mapsto x^p$ of $\mathbb{F}_p$.

Definition 2.5. For each $n \in \mathbb{N}^\times$, the *scaling endomorphism* $\varphi_n : \bar{\mathbb{N}} \to \bar{\mathbb{N}}$ is defined by

$$\varphi_n(x) := nx \quad (x \in \mathbb{N}), \qquad \varphi_n(+\infty) := +\infty,$$

where nx denotes ordinary multiplication in $\mathbb{N}$. Equivalently, $\varphi_n(x) = x^{\odot n}$, the n-fold tropical product of x with itself. In Lean, φ_n is defined as a semiring endomorphism `NBar ->+* NBar` via `scalingEndomorphism`.

Lemma 2.6. *For each $n \in \mathbb{N}^\times$, the map φ_n is a semiring endomorphism of $\bar{\mathbb{N}}$: it preserves both operations and both identities.*

Proof. Preservation of $+\infty$ and of tropical multiplication are immediate. Preservation of tropical addition uses $n \cdot \min(a, b) = \min(na, nb)$ for $n > 0$, which holds because scaling by n is order-preserving on $\mathbb{N} \cup \{+\infty\}$. In Lean this is the `map_add'` field of `scalingEndomorphism`, proved via monotonicity of `nsmul` and a case split on `le_total`. ☐

Lemma 2.7. *The scaling endomorphisms compose multiplicatively:*

$$\varphi_m \circ \varphi_n = \varphi_{mn}$$

for all $m, n \in \mathbb{N}^\times$. In particular, the map $n \mapsto \varphi_n$ is a monoid homomorphism from $(\mathbb{N}^\times, \times)$ into the semiring endomorphism monoid $\mathrm{End}_{\mathrm{Semiring}}(\bar{\mathbb{N}})$.

Proof. $(\varphi_m \circ \varphi_n)(x) = m(nx) = (mn)x = \varphi_{mn}(x)$. In Lean this is `scalingEndomorphism_comp`, proved via `pow_mul`. ☐

Lemma 2.8. *If $n > 1$, the fixed points of φ_n on $\bar{\mathbb{N}}$ are exactly the tropical zero $+\infty$ and tropical one 0.*

Proof. For a finite exponent x, the equation $nx = x$ with $n > 1$ forces $x = 0$; the point $+\infty$ is also fixed. Lean proves this by splitting `WithTop` $\mathbb{N}$ into its finite and top cases. ☐

Remark 2.9. The Frobenius *correspondences* $\Psi(\lambda)$, parametrised by $\lambda \in \mathbb{Q}_+^\times$, are a more refined object: they live in the *square* $\widehat{B\mathbb{N}^{\times}}^2$ of the Arithmetic Site and are defined via the action of $\mathbb{N}^\times \times \mathbb{N}^\times$ on $\bar{\mathbb{N}} \otimes_{\mathbb{B}} \bar{\mathbb{N}}$. They are introduced in Chapter 7.

Chapter 3

The presheaf topos $\widehat{\mathbb{N}^\times}$

The topos underlying the Arithmetic Site is the simplest kind: a presheaf topos. Its objects are sets equipped with a left $\mathbb{N}^\times$-action by arbitrary functions — not necessarily bijections, since $\mathbb{N}^\times$ is a monoid rather than a group — and its morphisms are $\mathbb{N}^\times$-equivariant maps between them. The multiplicative structure of $\mathbb{N}^\times$ is entirely encoded in the composition of these actions: if n and m act on a set X, then nm acts as the composite $x \mapsto n \cdot (m \cdot x)$.

Connes and Consani note that $\widehat{\mathbb{N}^\times} \simeq \mathrm{Sh}(\mathbb{N}^\times, J)$, where J is the *chaotic topology* on $\mathbb{N}^\times$ — the topology in which every presheaf is already a sheaf. This means there are no genuine gluing conditions to check: the topos is simply the presheaf category, with no further data. The covering families arising from factorizations of integers, foreshadowed in Chapter 1, reflect a finer site structure that one *can* put on $\mathbb{N}^\times$; the Arithmetic Site itself uses the coarser chaotic topology, where that combinatorial structure is not needed.

Definition 3.1. The *presheaf topos* $\widehat{\mathbb{N}^\times}$ is the category of functors $B{\mathbb{N}^\times}^{\mathrm{op}} \to \mathbf{Set}$, i.e. the category of sets equipped with a left $\mathbb{N}^\times$-action (using commutativity to exchange left and right actions). In Lean this is represented literally as the functor category `BNplusᵒᵖ ⥤ Type`.

Lemma 3.2. $\widehat{\mathbb{N}^\times}$ *has all small limits.*

Proof. This is the pointwise limit construction for a Type-valued functor category, supplied by Mathlib's `HasLimits` instance. □

Lemma 3.3. $\widehat{\mathbb{N}^\times}$ *has all small colimits.*

Proof. Colimits in a Type-valued functor category are computed pointwise. □

Lemma 3.4. $\widehat{\mathbb{N}^\times}$ *is cartesian closed.*

Proof. This is Mathlib's closed monoidal structure on a presheaf category. □

Lemma 3.5. $\widehat{\mathbb{N}^\times}$ *has a subobject classifier.*

Proof. Mathlib constructs the subobject classifier for presheaves. Together with finite limits and cartesian closure, this records the elementary topos structure in Lean; the standard general theorem says moreover that this presheaf category is a Grothendieck topos. □

Remark 3.6. The points of $\widehat{\mathbb{N}^\times}$ — geometric morphisms from **Set** to $\widehat{\mathbb{N}^\times}$ — are the deep link to number theory. Connes and Consani prove (Theorem 2.2 of their 2014 paper) that the category of points of $\widehat{\mathbb{N}^\times}$ is canonically equivalent to the category of totally ordered groups isomorphic to non-trivial subgroups of $(\mathbb{Q}, \mathbb{Q}_+)$. The space of isomorphism classes of such points is canonically isomorphic to the double quotient

$$\mathbb{Q}_+^\times \backslash \mathbb{A}^f / \hat{\mathbb{Z}}^*,$$

where $\mathbb{A}^f$ is the ring of finite adèles of $\mathbb{Q}$ and $\hat{\mathbb{Z}}^* = \hat{\mathbb{Z}}^\times$ is the group of units of the profinite completion of $\mathbb{Z}$. This quotient is a piece of the adèle class space of $\mathbb{Q}$, the same space that appears in Connes' noncommutative-geometric approach to the Riemann hypothesis. The structure sheaf $\mathcal{O} = \bar{\mathbb{N}}$ then equips this space with characteristic-1 geometry, whose Hasse–Weil zeta function recovers the Riemann zeta function.

Chapter 4

The structure sheaf

In classical algebraic geometry, a *ringed space* is a topological space equipped with a sheaf of commutative rings — the *structure sheaf* — whose stalks encode the local algebra of the space. The Arithmetic Site is a *ringed topos* in the same spirit: the role of the topological space is played by the presheaf topos $\widehat{\mathbb{N}^\times}$, and the role of the structure sheaf is played by $\bar{\mathbb{N}}$, viewed as a semiring object living inside that topos.

An object of $\widehat{\mathbb{N}^\times}$ is a set equipped with a left $\mathbb{N}^\times$-action, so for $\bar{\mathbb{N}}$ to be a semiring *object* of $\widehat{\mathbb{N}^\times}$ the action must be by semiring endomorphisms: each scaling map φ_n must preserve both tropical operations and both identities. This is exactly the content of Lemma 2.6, which is therefore the key structural fact underlying the definition below.

Definition 4.1. The *structure semiring* $\mathcal{O}$ of the Arithmetic Site is the commutative-semiring-valued presheaf whose value is $\bar{\mathbb{N}}$ and whose restriction maps are the scaling endomorphisms $n \cdot x := \varphi_n(x) = nx$.

Definition 4.2. The underlying Type-valued structure sheaf is obtained from $\mathcal{O}$ by forgetting its commutative-semiring structure pointwise.

Lemma 4.3. *The assignment $n \mapsto \varphi_n$ defines a monoid homomorphism $(\mathbb{N}^\times, \times) \to \mathrm{End}_{\mathrm{Semiring}}(\bar{\mathbb{N}})$, making $\bar{\mathbb{N}}$ into a semiring object of $\widehat{\mathbb{N}^\times}$ with $\mathbb{N}^\times$ acting by multiplication.*

Proof. Each φ_n is a semiring endomorphism by Lemma 2.6. The homomorphism property $\varphi_{mn} = \varphi_m \circ \varphi_n$ is Lemma 2.7, and $\varphi_1 = \mathrm{id}$ since $\varphi_1(x) = 1 \cdot x = x$. Lean packages both laws in `scalingMonoidHom`, using `scalingEndomorphism_comp` for multiplication. $\square$

Remark 4.4. Since the Arithmetic Site uses the chaotic topology, every presheaf on $B\mathbb{N}^\times$ is automatically a sheaf — there are no gluing conditions to verify. The content of the structure sheaf is therefore entirely algebraic: $\bar{\mathbb{N}}$ is not merely a set with $\mathbb{N}^\times$-action, but a semiring whose operations are *preserved* by that action. This is what Lemma 4.3 records.

Remark 4.5. Connes and Consani (Theorem 2.4 of their 2014 paper) describe the stalks of $\mathcal{O}$ explicitly. At the point corresponding to an intermediate semifield $F \subset K \subset \bar{F}$, the stalk is

$$\mathcal{O}_K = \{r \in K \mid r \oplus 1 = 1\},$$

the subsemiring of elements that are "bounded above by 1" in the ordering of K. This is the characteristic-1 analogue of the ring of integers of a local field: just as the ring of integers of a p-adic field K_v consists of elements of norm at most 1, the stalk of $\mathcal{O}$ picks out the elements "bounded above by 1". At the "generic" point (corresponding to $K = F = Z_{\max}$) the stalk recovers $\bar{\mathbb{N}}$ itself.

Chapter 5

The Arithmetic Site

All the ingredients are now in place. The presheaf topos $\widehat{\mathbb{N}^\times}$ provides the geometric stage. Connes–Consani identify its point classes with an adèlic quotient; the later chapter states precisely which part of that classification is currently formalized. The tropical semiring $\bar{\mathbb{N}}$, equipped with its $\mathbb{N}^\times$-action by scaling endomorphisms, is a semiring object of $\widehat{\mathbb{N}^\times}$: it plays the role of the ring of functions, recording characteristic-1 algebra at every point. Their combination is the Arithmetic Site.

Definition 5.1. The *Arithmetic Site* is the ringed topos $(\widehat{\mathbb{N}^\times}, \mathcal{O})$ consisting of the presheaf topos $\widehat{\mathbb{N}^\times}$ together with its structure sheaf $\mathcal{O} = \bar{\mathbb{N}}$. Since this Lean development fixes the ambient topos, `arithmeticSite` is the bundled commutative-semiring-valued presheaf on $(B\mathbb{N}^\times)^{\mathrm{op}}$; forgetting its algebraic structure gives `structureSheaf`.

Remark 5.2. The Arithmetic Site is a *ringed topos of characteristic* 1: its structure sheaf is a semiring with idempotent addition ($a \oplus a = a$), placing the entire construction in the world of tropical algebra rather than classical commutative ring theory. Connes and Consani describe it as the "algebraic geometric incarnation" of the non-commutative geometric approach to the Riemann hypothesis: the site provides a geometric framework, analogous to the function-field setting of the Weil proof, in which the relevant L-function — the Riemann zeta function — arises as a Hasse–Weil zeta function. The subsequent chapters make this precise by analysing the points of the site and the Frobenius correspondences acting on them.

Chapter 6

Points of the topos

In topos theory a *point* of a topos $\mathcal{E}$ generalises the notion of a point of a topological space: it is a geometric morphism $p : \mathbf{Set} \to \mathcal{E}$, pulling back each sheaf to its stalk. For a presheaf topos $\widehat{C}$, points correspond to flat functors $C \to \mathbf{Set}$. For $\widehat{\mathbb{N}^\times} = \widehat{B\mathbb{N}^\times}$ this means a point is represented by a flat $\mathbb{N}^\times$-action. The classification of these actions is a theorem of Connes–Consani. The present Lean development formalizes a flatness predicate and constructs the finite adèlic quotient occurring on the other side of that theorem; it does not yet formalize the classification equivalence itself.

Definition 6.1. A *point* of a topos $\mathcal{E}$ is a geometric morphism $p : \mathbf{Set} \to \mathcal{E}$, i.e. an adjoint pair $p^* \dashv p_*$ where p^* is left exact.

Definition 6.2. Let X carry an action of $\mathbb{N}^\times$. In the Lean development the proposition X *is a flat* $\mathbb{N}^\times$*-action* means that the category of elements of the associated type-valued action functor is cofiltered. This is the category-of-elements formulation of flatness used for the points of the presheaf topos.

Definition 6.3. A covariant $\mathbb{N}^\times$-action satisfies the *elementary flatness axioms* when its underlying set is nonempty, every two elements have a common predecessor under two positive-integer actions, and the action is free. Explicitly, the latter two conditions are

$$\forall x, y\ \exists z, m, n, \quad mz = x \ \text{ and } \ nz = y, \qquad mx = nx \Longrightarrow m = n.$$

Theorem 6.4. *For an* $\mathbb{N}^\times$*-action, the elementary axioms are equivalent to cofilteredness of its category of elements, and hence to the flatness predicate used for points of the Arithmetic Site.*

Proof. Nonemptiness and the common-predecessor axiom give the object cones in the definition of a cofiltered category. Freeness equalizes parallel arrows. Conversely, a cofiltered cone gives common predecessors, while a cofiltered equalizer and right cancellation in $\mathbb{N}^\times$ imply freeness. Both directions are formalized directly in Lean. □

Lemma 6.5. *The regular multiplication action of* $\mathbb{N}^\times$ *on itself is flat.*

Proof. The element 1 witnesses nonemptiness, the greatest common divisor of two positive integers supplies a common predecessor, and cancellation proves freeness. □

Remark 6.6. There is an important positive-part convention in the classification. If H is a non-trivial ordered subgroup of $(\mathbb{Q}, \mathbb{Q}_+)$, the associated flat action is on $H_{>0} = H \cap \mathbb{Q}_{>0}$, with

$n \cdot h = nh$; it is not the action on the whole additive group H. In particular, adjoining 0 to the action set would destroy the cancellation condition in the flatness axioms: for $n \neq m$ one would have $n \cdot 0 = m \cdot 0$.

6.1 The finite adèlic side

Definition 6.7. The type $\mathbb{A}^f$ of finite adèles of $\mathbb{Q}$ is the restricted product of the finite completions of $\mathbb{Q}$ with respect to their valuation subrings. In Lean this is Mathlib's finite adèle ring for the Dedekind domain $\mathbb{Z}$ and its fraction field $\mathbb{Q}$.

Definition 6.8. The profinite integers $\hat{\mathbb{Z}} \subset \mathbb{A}^f$ form the subring of finite adèles that are integral at every finite place. Its unit group is denoted $\hat{\mathbb{Z}}^*$.

Definition 6.9. The finite adèle class space used here is the orbit quotient

$$\mathbb{Q}_+^\times \backslash \mathbb{A}^f / \hat{\mathbb{Z}}^*.$$

Concretely, Lean combines the commuting actions of positive rational units and profinite-integer units into an action of their product and takes its orbit-relation quotient. At this stage this is a quotient type; no quotient topology is asserted.

Lemma 6.10. *Two finite adèles a, b have the same image in the formalized orbit quotient if and only if there is an element g of the combined scaling group such that $g \cdot b = a$.*

Definition 6.11. For $a \in \mathbb{A}^f$, define the additive subgroup

$$H_a := \{q \in \mathbb{Q} \mid qa \in \hat{\mathbb{Z}}\} \subseteq \mathbb{Q},$$

where q is embedded diagonally in the finite adèles. This subgroup, including its membership criterion at every finite place, is constructed in Lean.

Lemma 6.12. *If $u \in \hat{\mathbb{Z}}^*$ and $a \in \mathbb{A}^f$, then multiplication by u does not change the associated rational subgroup:*

$$H_{ua} = H_a.$$

Lemma 6.13. *If $r \in \mathbb{Q}_+^\times$, $a \in \mathbb{A}^f$, and $x \in \mathbb{Q}$, then*

$$x \in H_{ra} \iff rx \in H_a.$$

Together with Lemma 6.12, this is the formalized orbit-invariance calculation needed on the adèlic side of the published point classification.

Theorem 6.14. *The space of isomorphism classes of points of $\widehat{\mathbb{N}^\times}$ is canonically isomorphic to the double quotient*

$$\mathbb{Q}_+^\times \backslash \mathbb{A}^f / \hat{\mathbb{Z}}^*,$$

a piece of the adèle class space of $\mathbb{Q}$. More precisely, the category of points of $\widehat{\mathbb{N}^\times}$ is equivalent to the category of totally ordered groups isomorphic to non-trivial subgroups of $(\mathbb{Q}, \mathbb{Q}_+)$, with injective order-preserving homomorphisms as morphisms; the flat action associated to H uses $H_{>0}$ as above.

Proof. This is the published classification theorem of Connes–Consani (Theorem 2.2 in the 2014 note, and Theorem 2.1 with Proposition 2.5 in the full treatment). Their proof extracts a rank-one ordered group from a filtering action and reconstructs the action from its strictly positive part. The adèlic parametrization then sends a to H_a and identifies isomorphic ordered groups with the indicated double orbits.

Formalization status. Lean currently contains the flatness predicate, its elementary action-axiom characterization, the restricted-product adèle ring, $\hat{\mathbb{Z}}$, the orbit quotient, and $a \mapsto H_a$. It does not yet contain the functors in both directions or a proof that they induce the stated categorical equivalence and bijection on isomorphism classes. Consequently this published theorem intentionally has no Lean declaration link. $\square$

Chapter 7

The Frobenius correspondences

The Frobenius correspondences are the analogue, in characteristic 1, of the Frobenius endomorphisms of a variety over a finite field. They do not act on the Arithmetic Site itself — that would require endomorphisms of the ringed topos $(\widehat{\mathbb{N}^\times}, \mathcal{O})$ — but are constructed on its *square*. The algebra formalized so far is the raw tensor semiring $\bar{\mathbb{N}} \otimes_{\mathbb{B}} \bar{\mathbb{N}}$, where $\mathbb{B} = (\{0,1\}, \max, \times)$ is the Boolean semifield. There are two different levels below:

- tensor-algebra endomorphisms and rational evaluation maps, which are now implemented and proved in Lean; and
- reduced correspondences, their real parameter, their composition, and the exceptional tangential deformation, which remain published mathematics not yet represented by Lean structures.

7.1 The formalized tensor-algebra layer

Definition 7.1. The Boolean semifield $\mathbb{B}$ is realized in Lean as the bottom subsemiring of the idempotent semiring $\bar{\mathbb{N}}$. Its elements are the additive and multiplicative identities of $\bar{\mathbb{N}}$.

Definition 7.2. The raw structure semiring on the square is Mathlib's semiring tensor product

$$\bar{\mathbb{N}} \otimes_{\mathbb{B}} \bar{\mathbb{N}}.$$

Definition 7.3. Each scaling semiring endomorphism φ_n fixes the embedded Boolean semifield, and hence defines a $\mathbb{B}$-algebra endomorphism of $\bar{\mathbb{N}}$.

Definition 7.4. For $n, m \in \mathbb{N}^\times$, functoriality of the tensor product defines

$$\mathrm{Fr}_{n,m} : \bar{\mathbb{N}} \otimes_{\mathbb{B}} \bar{\mathbb{N}} \longrightarrow \bar{\mathbb{N}} \otimes_{\mathbb{B}} \bar{\mathbb{N}}, \qquad x \otimes y \longmapsto \varphi_n(x) \otimes \varphi_m(y).$$

Lemma 7.5. *On a pure tensor,*

$$\mathrm{Fr}_{n,m}(x \otimes y) = \varphi_n(x) \otimes \varphi_m(y).$$

Definition 7.6. Multiplication in $\bar{\mathbb{N}}$ induces the $\mathbb{B}$-algebra homomorphism

$$\mu : \bar{\mathbb{N}} \otimes_{\mathbb{B}} \bar{\mathbb{N}} \longrightarrow \bar{\mathbb{N}}, \qquad \mu(x \otimes y) = xy.$$

Lemma 7.7. *On pure tensors the product map satisfies $\mu(x \otimes y) = xy$.*

Definition 7.8. For $n, m \in \mathbb{N}^\times$, the *rational Frobenius evaluation* formalized in Lean is the algebra homomorphism

$$E_{n,m} := \mu \circ \mathrm{Fr}_{n,m} : \bar{\mathbb{N}} \otimes_{\mathbb{B}} \bar{\mathbb{N}} \longrightarrow \bar{\mathbb{N}}.$$

The Lean identifier retains the historical name `frobeniusCorrespondence`, but its type correctly records that it is an evaluation out of the tensor product, not an endomorphism of $\bar{\mathbb{N}}$ and not a reduced-correspondence triple. Lean does not impose coprimality on n, m; lowest terms matter only when the ratio n/m is used as a normalized rational parameter.

Lemma 7.9. *The rational evaluation has the expected pure-tensor formula*

$$E_{n,m}(x \otimes y) = \varphi_n(x)\varphi_m(y).$$

Theorem 7.10. *The tensor Frobenius endomorphisms compose multiplicatively in each index:*

$$\mathrm{Fr}_{n,m} \circ \mathrm{Fr}_{k,l} = \mathrm{Fr}_{nk,ml}.$$

Proof. The two algebra homomorphisms are equal by tensor-product extensionality. On the left and right generating factors the claim reduces respectively to $\varphi_n \circ \varphi_k = \varphi_{nk}$ and $\varphi_m \circ \varphi_l = \varphi_{ml}$, both already proved for the scaling endomorphisms. ☐

Remark 7.11. The theorem above is a theorem about an action on the raw tensor semiring. The maps $E_{n,m}$ have source $\bar{\mathbb{N}} \otimes_{\mathbb{B}} \bar{\mathbb{N}}$ and target $\bar{\mathbb{N}}$, so an expression such as $E_{n,m} \circ E_{k,l}$ is not even the relevant typed composition. It must not be confused with composition of the reduced correspondences described next.

7.2 Published reduced correspondences

For every *positive real* parameter $\lambda \in \mathbb{R}_+^\times$, Connes and Consani construct a semiring homomorphism

$$F(\lambda, q)\left(\sum_i q^{a_i} \otimes q^{b_i}\right) = q^{\inf_i(\lambda a_i + b_i)}$$

with a multiplicatively cancellative range semiring $R(\lambda)$. Thus the parameter in this formula is real, not merely rational. When $\lambda = n/m \in \mathbb{Q}_+^\times$ is in lowest terms, the integral-exponent map $E_{n,m}$ above is the raw rational evaluation underlying the construction; passing to the appropriate range and reduction is additional structure.

Definition 7.12. A *reduced correspondence* in the sense of Connes–Consani is a triple (R, ℓ, r), where R is a multiplicatively cancellative semiring and ℓ, r are semiring maps from the one-sided tropical semiring to R, each with zero kernel, whose two images generate R. Composition is defined from a tensor product over the middle semiring followed by multiplicatively cancellative reduction.

This structure, its reduction operation, and its composition are not yet defined in Lean in this project.

Definition 7.13. For $\lambda \in \mathbb{R}_+^\times$, the published *Frobenius correspondence* is

$$\Psi(\lambda) = \big(R(\lambda), \ell(\lambda), r(\lambda)\big),$$

obtained from $F(\lambda, q)$ and the two inclusions of the tensor factors. On geometric points over $\mathbb{R}_+^{\max}$ it corresponds to the positive-real scaling action on the associated adèle-class description. This geometric correspondence is not the ordinary $\mathbb{N}^\times$-scaling action on the global semiring, although that action is one algebraic ingredient in its rational models.

Theorem 7.14. *Let $\lambda, \lambda' \in \mathbb{R}_+^\times$. Composition of the published reduced Frobenius correspondences gives*

$$\Psi(\lambda) \circ \Psi(\lambda') = \Psi(\lambda\lambda')$$

when λ, λ' are both rational, and also whenever $\lambda\lambda'$ is irrational. If λ and λ' are both irrational but $\lambda\lambda'$ is rational, the composite is instead the canonical tangential deformation of $\Psi(\lambda\lambda')$.

Proof. This is Theorem 4.3 of the 2014 Connes–Consani note and Theorem 7.7 of the full paper. The proof computes the multiplicatively cancellative reduction of the tensor product of the range semirings $R(\lambda)$ and $R(\lambda')$. Flexibility present for irrational slopes survives the exceptional rational product and produces the tangential deformation.

Formalization status. Neither this reduction argument nor the irrational and tangential constructions are formalized. Accordingly, this theorem has no Lean declaration link; the proved theorem 7.10 is only its raw tensor-algebra precursor. □

Chapter 8

Connection to the Riemann zeta function

The connection with the zeta function has three logically distinct layers. First, standard analytic facts about the completed Riemann zeta function and its finite-prime logarithmic derivative are available in Mathlib and are proved in Lean. Second, Connes–Consani's published distributional trace formula identifies a geometric counting distribution for the scaling flow. Third, turning the explicit formula into the positivity needed for the Riemann hypothesis remains an open mathematical and formalization frontier. None of these layers should be conflated.

8.1 Analytic statements proved in Lean

Definition 8.1. The completed Riemann zeta function used by the project is Mathlib's meromorphically continued function completedRiemannZeta. Away from the poles of the Gamma factor it agrees with

$$\zeta_{\mathbb{Q}}(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s).$$

Using the continued function, rather than a totalized pointwise product, preserves the pole–zero cancellations at non-positive even integers.

Theorem 8.2. *The completed zeta function satisfies*

$$\zeta_{\mathbb{Q}}(1-s) = \zeta_{\mathbb{Q}}(s).$$

Theorem 8.3. *In Mathlib's normalization, the residue at $s = 1$ is 1:*

$$\lim_{s\to 1}(s-1)\zeta_{\mathbb{Q}}(s) = 1,$$

with the limit taken off the pole.

Theorem 8.4. *The completed zeta function is differentiable at every $s \neq 0, 1$.*

Definition 8.5. Define the finite-place Mellin transform to be the Dirichlet series of the von Mangoldt function,

$$M_f(s) := \sum_{n\geq 1} \frac{\Lambda(n)}{n^s}.$$

In Lean this is expressed using Mathlib's L-series. This definition records only finite-prime weights; it does not construct a fixed-point distribution on the Arithmetic Site.

Theorem 8.6. *If* $\mathrm{Re}(s) > 1$, *then*

$$M_f(s) = -\operatorname{logDeriv}(\zeta)(s) = -\frac{\zeta'(s)}{\zeta(s)}.$$

Proof. This is Mathlib's Dirichlet-series identity for the von Mangoldt function, rewritten using the project's definition. The half-plane hypothesis supplies convergence and excludes zeros and the pole from the quotient in question. □

Definition 8.7. The *completed analytic counting transform* is

$$M_{\mathrm{comp}}(s) := M_f(s) - \operatorname{logDeriv}(\Gamma_{\mathbb{R}})(s),$$

where $\Gamma_{\mathbb{R}}$ is Deligne's real Gamma factor. This adds the archimedean analytic contribution, but by definition alone it does not construct a counting distribution.

Theorem 8.8. *If* $\mathrm{Re}(s) > 1$, *then*

$$M_{\mathrm{comp}}(s) = -\operatorname{logDeriv}(\zeta_{\mathbb{Q}})(s).$$

Proof. On the right half-plane Lean proves $\zeta_{\mathbb{Q}} = \Gamma_{\mathbb{R}}\zeta$. The logarithmic derivative of this product, together with Theorem 8.6, gives the identity. □

Remark 8.9. Theorem 8.6 is the finite-place analytic term only; Theorem 8.8 adds the archimedean Gamma contribution and identifies the resulting analytic expression with the completed logarithmic derivative. Neither theorem constructs a distributional trace, a geometric fixed-point count, or the claimed Hasse–Weil interpretation. The functional equation by itself likewise gives no proof of the Riemann hypothesis.

8.2 The published distributional counting formula

Let $G = \mathbb{A}_{\mathbb{Q}}^{\times}/\mathbb{Q}^{\times}$ be the idèle class group and let $\vartheta_u\xi(x) = \xi(u^{-1}x)$ denote its scaling representation on the relevant quotient of the adèle class space. In the published trace formula, the geometric side is schematically

$$\mathrm{Tr}_{\mathrm{distr}}\left(\int_G h(u)\vartheta_u\, d^*u\right) = \sum_{v\in\Sigma_{\mathbb{Q}}} \int'_{\mathbb{Q}_v^{\times}} \frac{h(u^{-1})}{|1-u|_v}\, d^*u.$$

The prime is essential: the local expression at $u = 1$ is a normalized principal-value distribution, not an ordinary absolutely convergent integral. Haar measures, additive characters, and the normalization of the principal value must be fixed compatibly. The resulting "counting function" $N(u)$ is therefore a distribution paired with test functions, not a pointwise finite cardinality of fixed points.

Theorem 8.10. *With the normalizations of Connes–Consani, the zeta function* ζ_N *associated to the published counting distribution by the prescribed distributional Mellin relation*

$$\frac{\partial_s\zeta_N(s)}{\zeta_N(s)} = -\int_1^{\infty} N(u)u^{-s}\, d^*u$$

is the completed Riemann zeta function $\zeta_{\mathbb{Q}}(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$. *The integral notation denotes the distributional pairing and its stated regularization, not an ordinary integral of a pointwise function.*

Proof. This is the published counting-distribution result summarized as Theorem 2.7 of Connes–Consani (2014), using their preceding distributional trace formula and earlier computation of the counting distribution. Since a logarithmic derivative determines a meromorphic function only up to a nonzero multiplicative constant, the stated exact equality also uses their chosen normalization.

Formalization status. The project has not yet defined N as a distribution, formalized the principal-value local terms or the global trace formula, or proved that their Mellin transform equals `completeRiemannZeta`. Thus this published theorem deliberately has no Lean declaration link. Theorem 8.8 is a rigorous analytic bridge, not a construction or identification of the geometric distribution appearing here. □

8.3 The Riemann-hypothesis frontier

Definition 8.11. The project exposes `ArithmeticSite.RiemannHypothesis` as an alias for Mathlib's standard proposition asserting that the non-trivial zeros of the Riemann zeta function lie on the critical line. The Lean marker here certifies only that this target proposition is defined; it does not provide a term proving it.

Remark 8.12. No theorem in this repository proves the Riemann hypothesis, and the published Arithmetic-Site results quoted above do not themselves prove it. The present formal frontier consists of several substantial steps:

1. prove the classification of flat $\mathbb{N}^{\times}$-actions by ordered subgroups and finite adèle classes;

2. construct reduced correspondences, their cancellative composition, real slopes, and the tangential deformation;

3. build the relevant distribution spaces, normalized local principal values, and the global trace formula;

4. identify the resulting counting distribution with the completed zeta function, including the archimedean term and normalization;

5. establish the positivity statement in the explicit formula (or an adequate cohomological or intersection-theoretic substitute) that is equivalent to the Weil criterion for the Riemann hypothesis.

The current Lean results rigorously reach the tensor Frobenius action, the adèlic subgroup invariance calculations, and the completed analytic logarithmic derivative on $\mathrm{Re}(s) > 1$. The geometric distributional bridge and, above all, the final positivity mechanism remain unformalized and mathematically decisive.