

# Operator Algebras in Lean

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## 0.1 Introduction

## 0.2 Continuous functional calculus

**Definition 1** (Continuous functional calculus). A  $\ast$ - $R$ -algebra is said to have a continuous functional calculus for elements satisfying a predicate  $p$  if, for each  $a$  satisfying  $p$ , there is a  $\ast$ -homomorphism  $\phi_a : C(\sigma_R(a), R) \rightarrow A$  sending the identity function to  $a$ , and which is a closed embedding. Moreover,  $\sigma_R(a)$  is compact and nonempty, and  $\phi_a$  satisfies the spectral mapping property (i.e.,  $\sigma_R(\phi_a(f)) = f(\sigma_R(a))$ ).

**Definition 2** (Non-unital continuous functional calculus). A non-unital  $\ast$ - $R$ -algebra is said to have a non-unital continuous functional calculus for elements satisfying a predicate  $p$  if, for each  $a$  satisfying  $p$ , there is a non-unital  $\ast$ -homomorphism  $\phi_a : C(\sigma'_R(a), R)_0 \rightarrow A$  (here  $C(\sigma'_R(a), R)_0$  is the collection of functions vanishing at zero on the quasispectrum) sending the identity function to  $a$ , and which is a closed embedding. Moreover,  $\sigma'_R(a)$  is compact (it's always nonempty because it contains 0), and  $\phi_a$  satisfies the spectral mapping property (i.e.,  $\sigma'_R(\phi_a(f)) = f(\sigma'_R(a))$ ).

**Definition 3.** Given  $a \in A$  satisfying  $p$  and  $f : R \rightarrow R$  continuous on  $\sigma_R(a)$ , we define  $f(a) := \phi'_a(f)$  (and we give it a junk value of zero when either  $a$  does not satisfy  $p$  or  $f$  is not continuous on the spectrum).

**Definition 4.** Given  $a \in A$  satisfying  $p$  and  $f : R \rightarrow R$  continuous on  $\sigma_R(a)$  and  $f(0) = 0$ , we define  $f(a) := \phi'_a(f)$  (and we give it a junk value of zero when and of the conditions on  $a$  and  $f$  are not met).

**Theorem 5.** *For every normal element  $a$  in a unital  $C^\ast$ -algebra  $A$  there is a  $\ast$ -isomorphism between  $C(\sigma(a), \mathbb{C})$  and the  $C^\ast$ -subalgebra of  $A$  generated by  $a$ .*

*Proof.* Use the Gelfand transform. □

**Theorem 6.** *Every unital  $C^\ast$ -algebra has a continuous functional calculus for normal elements.*

*Proof.* Compose the  $\ast$ -isomorphism of Theorem 5, which is an isometry because it's an isomorphism of  $C^\ast$ -algebras, with the inclusion of  $C_1^\ast(a)$  (the unital  $C^\ast$ -subalgebra generated by  $a$ ) into  $A$ . The latter is also an isometry and therefore a closed embedding. □

**Theorem 7.** *Every unital  $\ast$ -algebra  $A$  with a continuous functional calculus for normal elements over  $\mathbb{C}$  has a continuous functional calculus for self-adjoint elements over  $\mathbb{R}$ .*

*Proof.* Since self-adjoint elements are normal, the continuous functional calculus for normal elements over  $\mathbb{C}$  with its spectral mapping property guarantees that the  $\mathbb{C}$ -spectrum of  $a \in A$  normal is actually contained in  $\mathbb{R}$ , and so coincides with the  $\mathbb{R}$ -spectrum of  $a$ . Therefore, the map which sends  $f \in C(\sigma_{\mathbb{R}}(a), \mathbb{R})$  to  $\hat{f} \in C(\sigma_{\mathbb{C}}(a), \mathbb{C})$  is a  $\ast$ -homomorphism, and composing it with  $\phi_a$  yields the desired  $\ast$ -homomorphism for the continuous functional calculus over  $\mathbb{R}$  for self-adjoint elements. □

**Theorem 8.** *Every unital  $\ast$ -algebra  $A$  which is a  $\ast$ -ordered ring (i.e., nonnegative elements are those of the form  $x^\ast x$ ) with the property that nonnegative elements have nonnegative spectrum, and with a continuous functional calculus for self-adjoint elements over  $\mathbb{R}$  has a continuous functional calculus for self-adjoint elements over  $\mathbb{R}_{\geq 0}$ .*

*Proof.* Omitted. □

## 0.3 Products of nonnegative elements are nonnegative

# Chapter 1

## Sakai's Book

### 1.1 Definitions of $C^*$ -algebras and $W^*$ -algebras

**Definition 9.** A Banach  $*$ -algebra  $\mathcal{A}$  is called a  $C^*$ -algebra if it satisfies  $\|x^*x\| = \|x\|^2$  for  $x \in \mathcal{A}$ .

**Definition 10.** A  $C^*$ -algebra  $\mathcal{M}$  is called a  $W^*$ -algebra if it is a dual space as a Banach Space (i.e., if there exists a Banach space  $\mathcal{M}_*$  such that  $(\mathcal{M}_*)^* = \mathcal{M}$ , where  $(\mathcal{M}_*)^*$  is the dual Banach space of  $\mathcal{M}_*$ ). We call such a Banach Space  $\mathcal{M}_*$  the predual of  $\mathcal{M}$ .